These questions are arranged roughly in the order the material was covered in class. Questions marked with a (*) are optional.

1. Let $G$ be the real numbers other than $-1$ and $\ast$ the operation $x \ast y = x + y + xy$.
   (a) Show that $G, \ast$ is a group.
   (b) Explicitly find its identity element and the inverse of an arbitrary element $x$.
   (c) Hence solve the equation $2 \ast x \ast 5 = 6$.

2. (a) Show that if a group has only one element of order 2, it must commute with the rest of the group. (Hint: if $h$ is this element, consider $ghg^{-1}$)
   (b) Show that a group cannot have exactly 2 elements of order 2 (Hint: $ghg^{-1}$, $gh$)

3. $S$ is a finite subgroup of $\mathbb{C}$, under multiplication. Show that $S$ is the set of roots of $z^n = 1$, for some $n \in \mathbb{N}$.

4. Let $\theta$ be a homomorphism $G \to H$. Show that the order of $\theta(g)$ divides the order of $g$, for any $g \in G$.

5. (*) Show that $(\mathbb{Z}, +)$ is not isomorphic to $(\mathbb{Q}, +)$.

6. (*) By considering generators and using Lagrange’s theorem, show that $S_3$ really is the only non-abelian group of order 6.

7. Consider a tetrahedron (a regular triangle-based pyramid). By applying the orbit-stabilizer theorem to one of its vertices, find:
   (a) The size of the group of rotation symmetries of the tetrahedron.
   (b) The size of the group of rotation and reflection symmetries of the tetrahedron.
   (c) What are these two groups? The answer to (b) is a clue.

8. Let $D$ be the “diagonal” subgroup of $G \times G$: $D = \{(g, g) | g \in G \}$. Show that $D$ is a normal subgroup if and only if $G$ is abelian.

9. (*) Let $G$ be an abelian group and $K \triangleleft G$ be the subgroup consisting of all elements of $G$ of finite order. Show that $G/K$ has no elements of finite order, other than the identity.

10. Show that $GL(n, \mathbb{R}) = SL(n, \mathbb{R}) \times \mathbb{R}^\times$. For what values of $n$ is this a direct product?
Group theory 2018 - question sheet 2: Lie groups

1. The Euclidean group, $E(n)$, can be represented in matrix form as $(n + 1) \times (n + 1)$ matrices, written schematically as

$$A(R, b) = \begin{pmatrix} R & b \\ 0 \ldots 0 & 1 \end{pmatrix}$$

Where $R$ is a $O(n)$ matrix and $b$ is a column vector in $\mathbb{R}^n$.
(a) Show that this group is indeed isomorphic to the Euclidean group.
(b) For $n = 2$, find a basis for the Lie algebra of this group.
(c) What are the commutation relations for this basis?

2. The Symplectic group, $Sp(2n, \mathbb{R})$ is the group of even-dimensional real square matrices, $S$ such that

$$SJS^T = J$$

(Note that this is the group of constant canonical transformations on a phase space)
(a) Show that the Lie algebra of $Sp(2n, \mathbb{R})$ is given by matrices $A$, such that $JA + A^T J = 0$.
(b) Hence show that

$$A = \begin{pmatrix} B & P \\ Q & -B^T \end{pmatrix},$$

where $B$ is an arbitrary $n \times n$ matrix and $P$ and $Q$ are symmetric.
(c) What is the condition $SJS^T = J$ for an arbitrary $2 \times 2$ matrix? Find a basis for $sp(2, \mathbb{R})$.

3. Consider the mapping $SU(2) \rightarrow SO(3)$ given in class: $A \rightarrow R$ such that

$$AX(x)A^\dagger = X(Rx)$$

Where, $x \in \mathbb{R}^3$, and $X(x)$ is the matrix given by $S_i x_i$ for some basis $S_i$ of $su(2)$.
Find the image of an element $A_i = e^{iS_i} \in SU(2)$ under this map, and hence show the map is surjective.

4. The Jacobi identity for Poisson brackets. Take three arbitrary continuous functions $F, G, H \in C(P)$. We may treat one of these functions as the “Hamiltonian” and write $\frac{d}{dt} = \{\cdot, H\}$. Consider the time evolution of $\{F, G\}$ and hence derive the Jacobi identity.

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1This is a real symplectic group, you may also see the term “symplectic group” used to refer to the complex symplectic group or to its compact subgroup - the symplectic matrices that are also unitary.
1. Let $G$ be a group with two representations $\Gamma_1$ and $\Gamma_2$ and let $H < G$ be a subgroup. The representations $\Gamma_i$ of $G$ define representations $\Gamma_i^H$ of $H$ given by $\Gamma_i^H(h) = \Gamma_i(h)$, $\forall h \in H$.
   a) Show that if $\Gamma_1$ and $\Gamma_2$ are equivalent, then so are the $\Gamma_i^H$.
   b) Does $\Gamma_i^H$ being irreducible imply that $\Gamma_i$ is?
   c) Does $\Gamma_i$ being irreducible imply that $\Gamma_i^H$ is?

2. In this problem we will see a representation of a group that is neither finite, nor a semi-simple Lie group:
   Consider the representation of $(\mathbb{R}, +)$ on $\mathbb{R}^2$ given by
   $$\Gamma(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$
   a) Show that this is a representation.
   b) Show that this representation is reducible.
   c) Show that this representation is not completely reducible.

3. Find all possible irreducible complex representations of $\mathbb{Z}_3$.

4. Consider the representation of $SU(2)$ on $C(\mathbb{C}^2)$, the space of differentiable functions in 2 complex variables:
   $$(\Gamma(A)f)(v) = f(A^{-1}v)$$
   for all functions $f \in C(\mathbb{C}^2)$, matrices $A \in SU(2)$ and $v = (z, w)^T \in \mathbb{C}^2$.
   a) Find the induced representation of $\mathfrak{su}(2)$ and write the operators $\gamma(S_i)$ as partial derivatives (hint: $e^{iS_i} = I \cos \frac{t}{2} + 2S_i \sin \frac{t}{2}$, in the standard basis for $\mathfrak{su}(2)$). You should find that one generator is $\gamma(S_i) = \frac{1}{2}(z \frac{\partial}{\partial w} - w \frac{\partial}{\partial z})$
   b) Consider the induced representation of $SU(2)$ on $P_l(\mathbb{C}^2)$, the space of rank-$l$ polynomials in 2 complex variables, which is a finite-dimensional subspace of $C(\mathbb{C}^2)$. Find a basis for $P_l(\mathbb{C}^2)$ that diagonalises one of the $\gamma(S_i)$; what are its eigenvalues? What irrep of $\mathfrak{su}(2)$ is this?
1. Show that the spin-1 (i.e. complex 3-vector) and adjoint representations of SU(2) are equivalent.

2. Consider the representation of \(su(2)\) on \(\Lambda^2(\mathbb{C}^2)\), the space of antisymmetric rank (2,0) complex tensors in 2 dimensions, induced by the fundamental (or spin-\(\frac{1}{2}\)) representation. Show that this is an irrep and find its spin.

3. **The addition of angular momenta and Clebsh-Gordan coefficients** Consider the tensor product of two spin-1 representations of \(su(2)\).
   (a) What irreducible representations are contained in the product representation?
   (b) The product contains a spin-1 irrep; what is its highest-weight vector in terms of the product basis \(\{|m_1,1\rangle|m_2,1\rangle\}\)?

4. A spin \(j\) particle decays into a spin \(k\) particle and a photon (spin 1). What are the possible values of \(k\) in terms of \(j\)?

5. What irreps of \(SO(3,1)\) are contained in the tensor product \((1,1) \otimes (1,1)\)? Arrange them into sets that are also irreps under parity.

6. **Isospin and baryons** In nuclear physics, the lightest quarks (the “up” and “down” quarks) are taken to transform under two separate \(su(2)\) symmetries:
   - The usual rotation (or “spin”) symmetry, under which each flavour of quark is a spin-\(\frac{1}{2}\) particle.
   - “Isospin”, an approximate \(SU(2)\) symmetry of strong force interactions, in which the up and down quarks span a spin-\(\frac{1}{2}\) representation space together. (sometimes known as “flavour” symmetry)

To begin with, we will only consider isospin. A **baryon** is a particle made from three quarks (the most familiar of these being the proton and neutron). The space of possible baryons made from the lightest two quarks will therefore transform under a tensor product of three spin-\(\frac{1}{2}\) isospin representations:
   a) How many orthogonal states will there be in this product?
   b) Arrange these baryons into irreps of isospin. (You should find three)
   As fermions, the total wavefunction of the quarks must be antisymmetric under the exchange of any two of the quarks. As these quarks transform under both spin and isospin independently, we are allowed to have the isospin part of the wavefunction either **antisymmetric** (if the spin part is symmetric) or **symmetric** (if the spin part is antisymmetric).
   c) What are the dimensions the spaces of totally symmetric and totally antisymmetric rank-3 tensors on \(\mathbb{C}^2\)? This is the number of baryons that we expect to observe.